Fixed Point Theorems on Parametric Metric Space and Parametric *b*- Metric Spaces

Umashankar Singh
Sagar Institute of Research & Technology Excellence, Bhopal, M.P, India.

Paresh Rakhe
Sagar Institute of Research & Technology Excellence, Bhopal, M.P, India.

Rakesh Shrivastva
Sagar Institute of Research & Technology, Bhopal, M.P, India.

Shyam Patkar
Bhabha Universirty, Bhopal, M.P, India.

Abstract: *In this Research paper we prove Some fixed point results on parametric metric spaces and parametric b-metric spaces which is generalization and extend the results of R.D.Daheriya[5] with new rational expression..*

Keywords: *Parametric metric spaces, parametric b- metric space, fixed point, common fixed point.*

I. INTRODUCTION:

The Concept of metric space has been generalized in many directions. In 2014, the study of expansive mapping is very important research area in fixed point theory. Hussain et. al.[9][10] introduced the new notion parametric metric spaces and proved some fixed point results. In 2014, Rao et. al.[15] present new type of generalized metric space and called parametric S-metric spaces and proved some common fixed point theorems under various expansive conditions. In 2015, Hussain at.al[10] and Krishnakumar et.al[11] extended this results and introduced Complete parametric b-metric spaces and prove some fixed point results with various examples. Later on, In 2016, Daheriya et.al[4][5] proved some fixed point theorem for continuous and surjective expansion type mapping. In 2018, Priyobarta et. al.[17] introduced parametric A-metric spaces and parametric N_b –Metric spaces and obtain some fixed point results and proved fixed – circle theorem for parametric N_b – Metric with an applications. In 2016 , R.D Daheriya [5] prove some fixed point theorem under various expansive conditions in parametric metric space and parametric b-metric spaces Now a day's fixed point theorems is used in various branches of mathematics, engineering, computer sciences ,economics, medical sciences and solve the problem of existence and uniqueness of various type types of problems for partial differential equations, Integral equations, approximate theory, and game theory etc.
In this Research paper, we prove some fixed point results on parametric metric spaces and parametric b-metric space and we generalized and extended the results of Daheriya at.al.[5], for new Rational expressions..

II. PRELIMINARIES:

Now we recall some definition and concept of parametric space for support our main results:

Definition 2.1[5] Let Xbe a Nonempty set and a function $\mathcal{P}: X \times X \times (0,\infty) \to [0,\infty)$is called Parametric Metric on X if

(a) $\mathcal{P}(x,y,t) = 0 \Rightarrow x = y\,, forall t > 0$
(b) $\mathcal{P}(x,y,t) = \mathcal{P}(y,x,t) forall t > 0$
(c) $\mathcal{P}(x,y,t) \le \mathcal{P}(x,z,t) + \mathcal{P}(z,y,t) forall x,y,z \in X and t > 0$

Then $(X,\mathcal{P})$ is called parametric metric space.

Definition 2.2[5] Let $\{x_n\}_{n=1}^{\infty}$is a sequence in parametric metric space $(X,\mathcal{P}) then$

(i) $\{x_n\}_{n=1}^{\infty}$is called convergent to $x \in X$ if $\lim_{n\to\infty}\mathcal{P}(x,x_n,t) = 0$ written as $\lim_{n\to\infty} x_n = x$ for all $t > 0$
(ii) $\{x_n\}_{n=1}^{\infty}$is called Cauchy sequence in X if $\lim_{n,m\to\infty}\mathcal{P}(x_n,x_m,t) = 0$ for all $t > 0$
(iii) A parametric metric space $(X,\mathcal{P})$ is called complete iff every Cauchy sequence is convergent to $x \in X$.

Definition 2.3[5] let $(X,\mathcal{P})$ be a complete parametric metric space and a function $\boldsymbol{T}: X \to X$ is a called continuous in $x \in X$ if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$ for all $t > 0$ then $\lim_{n\to\infty} \boldsymbol{T}\, x_n = \boldsymbol{T}x$.

Example: 2.4[5] Let $X = \{f/f: (0,+\infty) \to R\}$ and defined a function $\mathcal{P}: X \times X \times (0,\infty) \to [0,\infty)$ by $\mathcal{P}(g,k,t) = |g(t) - k(t)| forall\ g,k \in X and t > 0$ then $\mathcal{P}$is a parametric metric in X and the pair $(X,\mathcal{P})$ is called parametric metric space in X.

Definition 2.5[5] Let Xbe a Nonempty set and $b \ge 1$ be a real number and function $\mathcal{P}: X \times X \times (0,\infty) \to [0,\infty)$is called Parametric ***b***-Metric on X if

(d) $\mathcal{P}(x,y,t) = 0 \Rightarrow x = y\,, forall t > 0$
(e) $\mathcal{P}(x,y,t) = \mathcal{P}(y,x,t) forall t > 0$
(f) $\mathcal{P}(x,y,t) \le b[\mathcal{P}(x,z,t) + \mathcal{P}(z,y,t)] forall x,y,z \in X and t > 0\,, b \ge 1.$

and one says the pair $(X,\mathcal{P})$ is a parametric metric space with parameter $b \ge 1$. Obviously, for $b = 1$then Parametric b- metric space reduces into parametric metric space.

Definition 2.6[6] Let $\{x_n\}_{n=1}^{\infty}$is a sequence in parametric metric space $(X,\mathcal{P},b) then$

(iv) $\{x_n\}_{n=1}^{\infty}$is called convergent to $x \in X$ if $\lim_{n\to\infty}\mathcal{P}(x,x_n,t) = 0$ written as $\lim_{n\to\infty} x_n = x$ for all $t > 0$
(v) $\{x_n\}_{n=1}^{\infty}$is called Cauchy sequence in X if

$\lim_{n,m\to\infty}\mathcal{P}(x_n, x_m, t) = 0$ for all $t > 0$

(vi) A parametric b-metric space $(X, \mathcal{P}, b)$ is called complete iff every Cauchy sequence is convergent to $x \in X$.

Definition 2.7[6] let $(X, \mathcal{P}, b)$ be a complete parametric b-metric space and a function $\boldsymbol{T}: X \to X$ is a called continuous in $x \in X$ if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$ for all $t > 0$ then $\lim_{n\to\infty} \boldsymbol{T}\, x_n = \boldsymbol{T}x$.

Example: 2.8[5] Let $X = [0, +\infty)$ and defined a function $\mathcal{P}: X \times X \times (0, +\infty) \to [0, +\infty)$ by $\mathcal{P}(g, k, t) = t(g - k)^p \; for all \; g, k \in X and t > 0$ then $\mathcal{P}$ is a parametric b-metric in X with constants $b = 2^p$.

Lemma: 2.9 let $(X, \mathcal{P}, b)$ be a parametric b-metric space with the coefficient $b = 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. If $\{x_n\}_{n=1}^{\infty}$ converges to x and converges to y then $x = y$ then the limit of the sequence $\{x_n\}_{n=1}^{\infty}$ is unique.

Lemma: 2.10 let $(X, \mathcal{P}, b)$ be a parametric b-metric space with the coefficient $b = 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. such that

$\mathcal{P}(x_n, x_{n+1}, t) \le \beta \mathcal{P}(x_{n-1}, x_n, t)$, where $\beta \in \left[0, \frac{1}{b}\right)$ $and \; n = 1,2, \dots \dots$

Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(X, \mathcal{P}, b)$.

III. MAIN THEOREMS

[A] Fixed point theorem on Parametric metric space

$\boldsymbol{Theorem}[\mathbf{3.1}]$ *Let* $(X, \mathcal{P})$ *be a Complete Parametric Metric Space and* T *is a continuous mapping Satisfying the condition*:

$$\mathcal{P}(Tx, Ty, t) + \chi_1 \left\{\frac{\mathcal{P}(x, Ty, t)}{1 + \mathcal{P}(Tx, y, t)\mathcal{P}(y, Ty, t)}\right\} + \chi_2 \left\{\frac{\mathcal{P}(y, Ty, t)}{1 + \mathcal{P}(Tx, y, t)\mathcal{P}(y, Ty, t)}\right\} \le \chi_3 \mathcal{P}(x, y, t) + \chi_4 max\{\mathcal{P}(x, Tx, t), \mathcal{P}(y, Ty, t)\}$$

$for \; all \; x, y \in X \; , x \ne y \; , for \; all \; t > 0 \; and \; \chi_1, \chi_2, \chi_3, \chi_4 \ge 0,$

$and \; (\chi_3 + \chi_4 - 2\chi_1 - \chi_2) < 1 \;\; then \; T \; has \; a \; Fixed \; point \; in \; X.$

Proof: Choose $x_0 \in X$ is arbitrary and define a sequence $\{x_n\}_{n\in N}$ as follows $Tx_n = x_{n+1} \; for \; n = 1,2,3 \dots \dots$ then take $x = x_n \;\; and \; y = x_{n+1} \; then$

$$\mathcal{P}(Tx_n, Tx_{n+1}, t) + \chi_1 \left\{\frac{\mathcal{P}(x_n, Tx_{n+1}, t)}{1 + \mathcal{P}(Tx_n, x_{n+1}, t)\mathcal{P}(x_{n+1}, Tx_{n+1}, t)}\right\} + \chi_2 \left\{\frac{\mathcal{P}(x_{n+1}, Tx_{n+1}, t)}{1 + \mathcal{P}(Tx_n, x_{n+1}, t)\mathcal{P}(x_{n+1}, Tx_{n+1}, t)}\right\}$$

$$\le \chi_3 \mathcal{P}(x_n, x_{n+1}, t) + \chi_4 max\{\mathcal{P}(x_n, Tx_n, t), \mathcal{P}(x_{n+1}, Tx_{n+1}, t)\}$$

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) + \chi_1 \left\{\frac{\mathcal{P}(x_n, x_{n+2}, t)}{1 + \mathcal{P}(x_{n+1}, x_{n+1}, t)\mathcal{P}(x_{n+1}, x_{n+2}, t)}\right\} + \chi_2 \left\{\frac{\mathcal{P}(x_{n+1}, x_{n+2}, t)}{1 + \mathcal{P}(x_{n+1}, x_{n+1}, t)\mathcal{P}(x_{n+1}, x_{n+2}, t)}\right\}$$

$$\le \chi_3 \mathcal{P}(x_n, x_{n+1}, t) + \chi_4 max\{\mathcal{P}(x_n, x_{n+1}, t), \mathcal{P}(x_{n+1}, x_{n+2}, t)\}$$

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) + \chi_1 \mathcal{P}(x_n, x_{n+2}, t) + \chi_2 \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\le \chi_3 \mathcal{P}(x_n, x_{n+1}, t) + \chi_4 max\{\mathcal{P}(x_n, x_{n+1}, t), \mathcal{P}(x_{n+1}, x_{n+2}, t)\}$$

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) + \chi_1\{\mathcal{P}(x_n, x_{n+1}, t) + \mathcal{P}(x_{n+1}, x_{n+2}, t)\} + \chi_2 \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\le \chi_3 \mathcal{P}(x_n, x_{n+1}, t) + \chi_4 max\{\mathcal{P}(x_n, x_{n+1}, t), \mathcal{P}(x_{n+1}, x_{n+2}, t)\}$$

Case(I) if $max\{\mathcal{P}(x_n, x_{n+1}, t), \mathcal{P}(x_{n+1}, x_{n+2}, t)\} = \mathcal{P}(x_n, x_{n+1}, t) \; then \; we \; get$

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) + \chi_1\{\mathcal{P}(x_n, x_{n+1}, t) + \mathcal{P}(x_{n+1}, x_{n+2}, t)\} + \chi_2 \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\le \chi_3 \mathcal{P}(x_n, x_{n+1}, t) + \chi_4 \mathcal{P}(x_n, x_{n+1}, t)$$

$$(1 + \chi_1 + \chi_2)\mathcal{P}(x_{n+1}, x_{n+2}, t) \le (\chi_3 + \chi_4 - \chi_1)\mathcal{P}(x_n, x_{n+1}, t)$$

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) \le \frac{(\hbar_3 + \hbar_4 - \hbar_1)}{(1 + \hbar_1 + \hbar_2)} \mathcal{P}(x_n, x_{n+1}, t)$$

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) \le \beta \, \mathcal{P}(x_n, x_{n+1}, t) \, , \quad Where \; \beta = \frac{(\chi_3 + \chi_4 - \chi_1)}{(1 + \chi_1 + \chi_2)} < 1$$

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) \le \beta^2 \mathcal{P}(x_{n-1}, x_n, t)$$

……

……

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) \le \beta^n \mathcal{P}(x_0, x_1, t)$$

By lemma (2.5) ,$\{x_n\}_{n\in N}$is a Cauchy sequence in X .But X be a Complete Parametric space. Hence the sequence $\{x_n\}_{n\in N}$ is converges. Call the limit $\acute{\alpha} \in X$. $then\ x_n \to \acute{\alpha}\ \ as\ n \to +\infty$. By continuity of $\boldsymbol{T}\ \ we\ have$,

$$T\acute{\alpha} = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = \acute{\alpha}$$

That is , $T\acute{\alpha} = \acute{\alpha}$; thus , T has a fixed point of X.

Case(II)if $max\{\mathcal{P}(x_n, x_{n+1}, t), \mathcal{P}(x_{n+1}, x_{n+2}, t)\} = \mathcal{P}(x_{n+1}, x_{n+2}, t)\ then\ we\ get$

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) + \chi_1\{\mathcal{P}(x_n, x_{n+1}, t) + \mathcal{P}xn{+}1{,}xn{+}2{,}t{+}\chi 2\mathcal{P}xn{+}1{,}xn{+}2{,}t$$

$$\leq \chi_3\mathcal{P}(x_n, x_{n+1}, t) + \chi_4\mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$(1 + \chi_1 + \chi_2 - \chi_4)\mathcal{P}(x_{n+1}, x_{n+2}, t) \leq (\chi_3 - \chi_1)\mathcal{P}(x_n, x_{n+1}, t)$$

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) \leq \frac{(\chi_3 - \chi_1)}{(1 + \chi_1 + \chi_2 - \chi_4)}\mathcal{P}(x_n, x_{n+1}, t)$$

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) \leq \beta\ \mathcal{P}(x_n, x_{n+1}, t)\ \ where\ \ \beta = \frac{(\chi_3 - \chi_1)}{(1 + \chi_1 + \chi_2 - \chi_4)} < 1$$

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) \leq \beta^2\mathcal{P}(x_{n-1}, x_n, t)$$

……..

……

$\mathcal{P}(x_{n+1}, x_{n+2}, t) \leq \beta^n\mathcal{P}(x_0, x_1, t)$By lemma (2.5) ,$\{x_n\}_{n\in N}$is a Cauchy sequence in X .But X be a Complete Parametric Metric space. Hence the sequence $\{x_n\}_{n\in N}$ is converges. Call the limit $\acute{\alpha} \in X$.

$then\ x_n \to \acute{\alpha}\ \ as\ n \to +\infty$. By continuity of $\boldsymbol{T}\ \ we\ have$,

$$T\acute{\alpha} = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = \acute{\alpha}$$

That is, $T\acute{\alpha} = \acute{\alpha}$; thus, $\boldsymbol{T}$ has a fixed point of X.

Uniqueness: Suppose $\acute{\acute{\alpha}}$ be another fixed point of T in X then $T\acute{\acute{\alpha}} = \acute{\acute{\alpha}}\ \ and\ \ T\acute{\alpha} = \acute{\alpha}$. Now,

$$\mathcal{P}(\boldsymbol{T}\acute{\alpha}, \boldsymbol{T}\acute{\acute{\alpha}}, t) + \chi_1\left\{\frac{\mathcal{P}(\acute{\alpha}, \boldsymbol{T}\acute{\acute{\alpha}}, t)}{1 + \mathcal{P}(\boldsymbol{T}\acute{\alpha}, \acute{\acute{\alpha}}, t)\mathcal{P}(\acute{\acute{\alpha}}, \boldsymbol{T}\acute{\acute{\alpha}}, t)}\right\} + \chi_2\left\{\frac{\mathcal{P}(\acute{\acute{\alpha}}, \boldsymbol{T}\acute{\acute{\alpha}}, t)}{1 + \mathcal{P}(\boldsymbol{T}\acute{\alpha}, \acute{\acute{\alpha}}, t)\mathcal{P}(\acute{\acute{\alpha}}, \boldsymbol{T}\acute{\acute{\alpha}}, t)}\right\} \leq \chi_3\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t) + \chi_4 max\{\mathcal{P}(\acute{\alpha}, \boldsymbol{T}\acute{\alpha}, t), \mathcal{P}(\acute{\acute{\alpha}}, \boldsymbol{T}\acute{\acute{\alpha}}, t)\}$$

$$\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t) + \chi_1\left\{\frac{\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t)}{1 + \mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t)\mathcal{P}(\acute{\acute{\alpha}}, \acute{\acute{\alpha}}, t)}\right\} + \chi_2\left\{\frac{\mathcal{P}(\acute{\acute{\alpha}}, \acute{\acute{\alpha}}, t)}{1 + \mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t)\mathcal{P}(\acute{\acute{\alpha}}, \acute{\acute{\alpha}}, t)}\right\} \leq \chi_3\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t) + \chi_4 max\{\mathcal{P}(\acute{\alpha}, \acute{\alpha}, t), \mathcal{P}(\acute{\acute{\alpha}}, \acute{\acute{\alpha}}, t)\}$$

$$\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t) + \chi_1\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t) \leq \chi_3\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t)$$

$$\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t) \leq (\hbar_3 - \hbar_1)\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t)$$

$$\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t) \leq \frac{1}{(\chi_1 - \chi_3)}\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t)$$

This is only true if $\mathcal{P}(\acute{\alpha}, \acute{\acute{\alpha}}, t) = 0 \Rightarrow \acute{\alpha} = \acute{\acute{\alpha}}$.

$Hence\ \boldsymbol{T}\ has\ a\ unique\ Fixed\ point\ in\ X.$

[B] Fixed point theorem on parametric b-metric space

$\boldsymbol{Theorem}[\mathbf{3.2}]\ Let\ (X, \mathcal{P}, b) be\ a\ Complete\ Parametric\ Metric\ Space\ and\ \mathfrak{T}\ is\ a\ continuous\ mapping\ Satisfying\ the\ condition$:

$$\mathcal{P}(\boldsymbol{T}x, \boldsymbol{T}y, t) + \chi_1\left\{\frac{\mathcal{P}(x, \boldsymbol{T}y, t)}{1 + \mathcal{P}(\boldsymbol{T}x, y, t)\mathcal{P}(y, \boldsymbol{T}y, t)}\right\} + \chi_2\left\{\frac{\mathcal{P}(y, \boldsymbol{T}y, t)}{1 + \mathcal{P}(\boldsymbol{T}x, y, t)\mathcal{P}(y, \boldsymbol{T}y, t)}\right\} \leq \chi_3\mathcal{P}(x, y, t) + \chi_4 max\{\mathcal{P}(x, \boldsymbol{T}x, t), \mathcal{P}(y, \boldsymbol{T}y, t)\}$$

$for\ all\ x, y \in X\ , x \neq y\ , for\ all\ t > 0\ and\ \chi_1, \chi_2, \chi_3, \chi_4 \geq 0,$

$and\ \ b(\chi_3 + \chi_4) - b(1 + b)\chi_1 - \chi_2) < 1\ \ then\ \mathfrak{T}\ has\ a\ Fixed\ point\ in\ X.$

Proof: Choose $x_0 \in X$ is arbitrary and define a sequence$\{x_n\}_{n\in N}$ as follows $\boldsymbol{T}x_n = x_{n+1}\ for\ n =$ 1,2,3 … … then take $x = x_n\ \ and\ y = x_{n+1}\ then$

$$\mathcal{P}(\boldsymbol{T}x_n, \boldsymbol{T}x_{n+1}, t) + \chi_1\left\{\frac{\mathcal{P}(x_n, \boldsymbol{T}x_{n+1}, t)}{1 + \mathcal{P}(\boldsymbol{T}x_n, x_{n+1}, t)\mathcal{P}(x_{n+1}, \boldsymbol{T}x_{n+1}, t)}\right\} + \chi_2\left\{\frac{\mathcal{P}(x_{n+1}, \boldsymbol{T}x_{n+1}, t)}{1 + \mathcal{P}(\boldsymbol{T}x_n, x_{n+1}, t)\mathcal{P}(x_{n+1}, \boldsymbol{T}x_{n+1}, t)}\right\}$$

$$\leq \chi_3\mathcal{P}(x_n, x_{n+1}, t) + \chi_4 max\{\mathcal{P}(x_n, \boldsymbol{T}x_n, t), \mathcal{P}(x_{n+1}, \boldsymbol{T}x_{n+1}, t)\}$$

Published By:
International Journal of Engineering Research in Current Trends

$$\mathcal{P}(x_{n+1},x_{n+2},t) + \chi_1\left\{\frac{\mathcal{P}(x_n,x_{n+2},t)}{1+\mathcal{P}(x_{n+1},x_{n+1},t)\mathcal{P}(x_{n+1},x_{n+2},t)}\right\} + \chi_2\left\{\frac{\mathcal{P}(x_{n+1},x_{n+2},t)}{1+\mathcal{P}(x_{n+1},x_{n+1},t)\mathcal{P}(x_{n+1},x_{n+2},t)}\right\}$$

$$\leq \chi_3\mathcal{P}(x_n,x_{n+1},t) + \chi_4 max\{\mathcal{P}(x_n,x_{n+1},t),\mathcal{P}(x_{n+1},x_{n+2},t)\}$$

$$\mathcal{P}(x_{n+1},x_{n+2},t) + \chi_1\mathcal{P}(x_n,x_{n+2},t) + \chi_2\mathcal{P}(x_{n+1},x_{n+2},t)$$

$$\leq \chi_3\mathcal{P}(x_n,x_{n+1},t) + \chi_4 max\{\mathcal{P}(x_n,x_{n+1},t),\mathcal{P}(x_{n+1},x_{n+2},t)\}$$

$$\mathcal{P}(x_{n+1},x_{n+2},t) + b\chi_1[\mathcal{P}(x_n,x_{n+1},t) + \mathcal{P}(x_{n+1},x_{n+2},t)] + \chi_2\mathcal{P}(x_{n+1},x_{n+2},t)$$

$$\leq \chi_3\mathcal{P}(x_n,x_{n+1},t) + \chi_4 max\{\mathcal{P}(x_n,x_{n+1},t),\mathcal{P}(x_{n+1},x_{n+2},t)\}$$

Case(I) if $max\{\mathcal{P}(x_n,x_{n+1},t),\mathcal{P}(x_{n+1},x_{n+2},t)\} = \mathcal{P}(x_n,x_{n+1},t)$ *then we get*

$$\mathcal{P}(x_{n+1},x_{n+2},t) + b\chi_1[\mathcal{P}(x_n,x_{n+1},t) + \mathcal{P}(x_{n+1},x_{n+2},t)] + \chi_2\mathcal{P}(x_{n+1},x_{n+2},t)$$

$$\leq \chi_3\mathcal{P}(x_n,x_{n+1},t) + \chi_4\mathcal{P}(x_n,x_{n+1},t)$$

$$(1+b\chi_1+\chi_2)\mathcal{P}(x_{n+1},x_{n+2},t) \leq (\chi_3+\chi_4-b\chi_1)\mathcal{P}(x_n,x_{n+1},t)$$

$$\mathcal{P}(x_{n+1},x_{n+2},t) \leq \frac{(\chi_3+\chi_4-b\chi_1)}{(1+b\chi_1+\chi_2)}\mathcal{P}(x_n,x_{n+1},t)$$

$$\mathcal{P}(x_{n+1},x_{n+2},t) \leq \beta\,\mathcal{P}(x_n,x_{n+1},t)\,,\quad Where\ \beta = \frac{(\chi_3+\chi_4-b\chi_1)}{(1+b\chi_1+\chi_2)} < \frac{1}{b}$$

Hence by induction we get,

$$\mathcal{P}(x_{n+1},x_{n+2},t) \leq \beta^{n+1}\mathcal{P}(x_0,x_1,t)$$

By lemma (2.10),$\{x_n\}_{n\in N}$is a Cauchy sequence in X .But X be a Complete Parametric b-Metric space. Hence the sequence $\{x_n\}_{n\in N}$ is converges. Call the limit $\acute{\alpha} \in X$. *then* $x_n \to \acute{\alpha}$ *as* $n \to +\infty$. By continuity of $\boldsymbol{T}$ *we have* ,

$$\boldsymbol{T}\acute{\alpha} = \boldsymbol{T}(\lim_{n\to\infty} x_n) = \lim_{n\to\infty}\boldsymbol{T}x_n = \lim_{n\to\infty} x_{n+1} = \acute{\alpha}$$

That is , $\boldsymbol{T}\acute{\alpha} = \acute{\alpha}$; thus , $\boldsymbol{T}$ has a fixed point of X.

Case(II) if $max\{\mathcal{P}(x_n,x_{n+1},t),\mathcal{P}(x_{n+1},x_{n+2},t)\} = \mathcal{P}(x_{n+1},x_{n+2},t)$ *then we get*

$$\mathcal{P}(x_{n+1},x_{n+2},t) + b\chi_1[\mathcal{P}(x_n,x_{n+1},t) + \mathcal{P}(x_{n+1},x_{n+2},t)] + \chi_2\mathcal{P}(x_{n+1},x_{n+2},t)$$

$$\leq \chi_3\mathcal{P}(x_n,x_{n+1},t) + \chi_4\mathcal{P}(x_{n+1},x_{n+2},t)$$

$$(1+b\chi_1+\chi_2-\chi_4)\mathcal{P}(x_{n+1},x_{n+2},t) \leq (\chi_3-b\chi_1)\mathcal{P}(x_n,x_{n+1},t)$$

$$\mathcal{P}(x_{n+1},x_{n+2},t) \leq \frac{(\chi_3-b\chi_1)}{(1+b\chi_1+\chi_2-\chi_4)}\mathcal{P}(x_n,x_{n+1},t)$$

$$\mathcal{P}(x_{n+1},x_{n+2},t) \leq \beta\,\mathcal{P}(x_n,x_{n+1},t)\ \ where\ \beta = \frac{(\chi_3-b\chi_1)}{(1+b\chi_1+\chi_2-\chi_4)} < \frac{1}{b}$$

Hence by induction we get,

$$\mathcal{P}(x_{n+1},x_{n+2},t) \leq \beta^{n+1}\mathcal{P}(x_0,x_1,t)$$

By lemma (2.10) ,$\{x_n\}_{n\in N}$is a Cauchy sequence in X .But X be a Complete Parametric space. Hence the sequence $\{x_n\}_{n\in N}$ is converges. Call the limit $\acute{\alpha} \in X$.

then $x_n \to \acute{\alpha}$ *as* $n \to +\infty$. By continuity of$\boldsymbol{T}$ *we have* ,

$$\boldsymbol{T}\acute{\alpha} = \boldsymbol{T}(\lim_{n\to\infty} x_n) = \lim_{n\to\infty}\boldsymbol{T}x_n = \lim_{n\to\infty} x_{n+1} = \acute{\alpha}$$

That is, $\boldsymbol{T}\acute{\alpha} = \acute{\alpha}$; thus, $\boldsymbol{T}$ has a fixed point of X.

Uniqueness: Suppose $\acute{\acute{\alpha}}$ be another fixed point of $\boldsymbol{T}$ in X then $\boldsymbol{T}\acute{\acute{\alpha}} = \acute{\acute{\alpha}}$ *and* $\boldsymbol{T}\acute{\alpha} = \acute{\alpha}$. Now,

$$\mathcal{P}(\boldsymbol{T}\acute{\alpha},\boldsymbol{T}\acute{\acute{\alpha}},t) + \chi_1\left\{\frac{\mathcal{P}(\acute{\alpha},\boldsymbol{T}\acute{\acute{\alpha}},t)}{1+\mathcal{P}(\boldsymbol{T}\acute{\alpha},\acute{\acute{\alpha}},t)\mathcal{P}(\acute{\acute{\alpha}},\boldsymbol{T}\acute{\acute{\alpha}},t)}\right\} + \chi_2\left\{\frac{\mathcal{P}(\acute{\acute{\alpha}},\boldsymbol{T}\acute{\acute{\alpha}},t)}{1+\mathcal{P}(\boldsymbol{T}\acute{\alpha},\acute{\acute{\alpha}},t)\mathcal{P}(\acute{\acute{\alpha}},\boldsymbol{T}\acute{\acute{\alpha}},t)}\right\} \leq \chi_3\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t) + \chi_4 max\{\mathcal{P}(\acute{\alpha},\boldsymbol{T}\acute{\alpha},t),\mathcal{P}(\acute{\acute{\alpha}},\boldsymbol{T}\acute{\acute{\alpha}},t)\}$$

$$\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t) + \chi_1\left\{\frac{\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t)}{1+\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t)\mathcal{P}(\acute{\acute{\alpha}},\acute{\acute{\alpha}},t)}\right\} + \chi_2\left\{\frac{\mathcal{P}(\acute{\acute{\alpha}},\acute{\acute{\alpha}},t)}{1+\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t)\mathcal{P}(\acute{\acute{\alpha}},\acute{\acute{\alpha}},t)}\right\} \leq \chi_3\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t) + \chi_4 max\{\mathcal{P}(\acute{\alpha},\acute{\alpha},t),\mathcal{P}(\acute{\acute{\alpha}},\acute{\acute{\alpha}},t)\}$$

$$\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t) + \chi_1\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t) \leq \chi_3\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t)$$

$$\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t) \leq (\chi_3-\chi_1)\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t)$$

$$\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t) \leq \frac{1}{(\chi_1-\chi_3)}\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t)$$

This is only true if $\mathcal{P}(\acute{\alpha},\acute{\acute{\alpha}},t) = 0 \Rightarrow \acute{\alpha} = \acute{\acute{\alpha}}$.

Hence $\boldsymbol{T}$ *has a unique Fixed point in* X.

REFERENCES

1. Ansari A.H, Kumar J., Vashistha S., C-class function on Common Fixed point theorem of Weakly Compatible Maps in Partial Metric space. Int. J. of Adv. In math., 2019,3, pp-15-23.
2. Ansari A.H., Note on $\varphi - \phi$-contractive type mapping and related fixed point theorem. In the 2nd Regional conference on Math. and Appl., Payame Noor Univ., Tehran, Iran, 2014, pp-377-380.
3. Choudhary R., Garg A.K., Fixed point results in parametric metric space., Int. J. on Emerging Tech., 10(2b) , 2019, 100-104.
4. Daheriya R.D, Jain R., Ughade M., Fixed point, coincidence point and common fixed point theorems under various expansive conditions in parametric metric space and parametric b-metric spaces. Gazi. Univ. J., 2016, 29 , 95-107.
5. Daheriya R.D, Shrivastva S., Ughade M., Parametric metric space, parametric b-metric space and expansive typing mapping. Int. J. Math. App. 2016, 4,107-117.
6. Ege Ozgur, De la Sen Manuel, A new perspective on Parametric Metric spaces. Σ-mathematics, 2019.7, 1008.
7. Ege, O., karaca, I. , Fixed point theorems and an application in parametric metric space ., Azerb. J. Math. 2017, 7, 27-39.
8. George A. and Veeramani P., On some results in fuzzy metric space, fuzzy sets and systems, 64(3) 1994, 395399.
9. Hussain N, Khaleghizadeh S., Salimi P. , Abdou A.A.N., A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces. Abstr.Appl. anal. 2014, doi: 10.1155/2014/690139.
10. Hussain N. , Salimi, P., Parvaneh, V., Fixed point results for various contractions in parametric and fuzzy b-metric spaces., J. Nonlinear Sci. Appl, 2015,719-739.
11. Krishnakumar R.; Sanaatammappa, N.P. ,Some fixed point theorem in parametric metric space., Int. J. Math. Res. Sci. , 2016, 8, 213-220.
12. Krishnakumar, R.; Sanaatammappa, N.P., Some fixed point theorem in parametric b-metric space., Int. J. Math. Sci. Eng. Appl, 2016, 10, 99-106
13. Likhitkar S., Daheriya R.D., Ughade M., Common fixed point theorems in parametric metric space under nonlinear type contractions. Int.J.Math. Arch. 2016, 7, 105-109.
14. Posul H., Kukukeu S., On parametric spaces. Bull. Math.Stat. Res., 2017, 5, 17-20.
15. Rao, K.P.R; Babu, D.V;Ramudu, E.T. , Some unique common fixed point theorem in parametric s-metric space . Int. J. Innev. Res. Sci. Eng. Technol 2014, 3, 14375-14387.
16. Shatanawi W., Coupled fixed point theorem in generalized metric spaces. Hacet. J.Math. Stat. 2011, 40, 441-447.
17. Tas N., Ozgur N.Y., On parametric S-metric spaces and fixed point type theorems for expansive mapping. J.Math. 2016. Dol:10.1155/2016/4746732.
18. Tas, N. ,Ozgur, N.Y, some fixed point results on parametric N_b-metric space . , Commun . korean Math. Soc., 2018, 33, 943-960.